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The structure of quotients of the Onsager algebra by closed ideals*

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Abstract. We study the Onsager algebra from the ideal theoretic point of view. A complete classification of closed ideals and the structure of quotient algebras are obtained. We also discuss the solvable algebra aspect of the Onsager algebra through the use of formal Lie algebras.

1. Introduction

By the Onsager algebra we mean an infinite-dimensional Lie algebra with a basis A_m, G_m ($m \in \mathbb{Z}$) and the commutation relations:

$$[A_m, A_l] = 4G_{m-l} \tag{1a}$$

$$[A_m, G_l,] = 2A_{m-l} - 2A_{m+l} \tag{1b}$$

$$[G_m, G_l] = 0. \tag{1c}$$

This Lie algebra appeared in the seminal paper of Onsager [19], in which the free energy of the two-dimensional Ising model was computed exactly. We shall call this algebra the Onsager algebra following the convention in [9]. In his paper Onsager exploited the (row-to-row) transfer matrix method, by which the calculation amounts to the calculation of the largest eigenvalue of the transfer matrix. The transfer matrix has the form

$$\text{constant} \times \mathrm{e}^A \mathrm{e}^B$$

where A and B are matrices of degree 2^n (n being the number of sites on a row), which are the linear sums of tensor products of Pauli matrices. By analysing the structure of the algebra (representation) generated by A and B in detail, Onsager derives the algebra (1) or its representations. The number of sites n is used in the representation or in the structure of quotient algebra as follows:

$$A_{m+2n} = A_m \qquad G_{m+2n} = G_m.$$

Although the structure changes slightly depending on the parity of n, the resulting representation is a direct sum of quaternions and scalars. Utilizing this analysis Onsager computed the largest, the second largest eigenvalues and corresponding eigenvectors of the

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transfer matrix. Later, Onsager re-solved the two-dimensional Ising model by using the now famous free fermions (Clifford algebras) with Kaufman [14, 15]. The method of the free fermion is a much more powerful one than that based on the algebra (1). This might be the reason why the Onsager algebra was forgotten for a while. In the 1980s, this algebra reappeared with the renewed interest in two-dimensional integrable field theories. Dolan and Grady [11] rediscovered this algebra while studying the condition for a two-dimensional field theory to possess an infinite number of conservation laws. Subsequently this algebra again appeared in the study of integrable spin chains [12], and then in the superintegrable chiral Potts model [4]. The spectrum of the superintegrable chiral Potts model is shown to obey certain quadratic equations by a numerical study in [3]. Davies [9] studied representations of the quotient of the Onsager algebra by an ideal generated by a linear relation among A_js and gave an answer to this observation. Except for these, not so much attention was paid on the Onsager algebra. One of the present authors [23] found that the Onsager algebra can be presentated as an invariant subalgebra of the loop algebra $\mathbb{C}[t, t^{-1}] \otimes \mathfrak{sl}_2$ by an involution while examining the papers [9, 10]. A generalization of the Onsager algebra to the case sl_n was studied in [25], which enhances the work [1]. Although the paper [9] is full of inspiring content, it contains somewhat ambiguous settings and claims from a mathematical viewpoint. This was one of the motivations for the paper [23] and also the present one.

The present study aims to pursue the direction in [9, 10], but with more stress on the quotient algebras by ideals rather than the representations as in [9, 23]. In the course of our study we focus our attention on the case when the quotient algebras do not have the centre. Such an ideal whose quotient algebra does not contain central elements will be called a closed ideal in this paper. We classify all the closed ideals of the Onsager algebra by exploiting the presentation of this algebra as a subalgebra of the loop algebra mentioned above. To each closed ideal there corresponds a 'reciprocal' polynomial in one variable. The structure of the quotient algebra differs according to whether the corresponding polynomial has ± 1 as its zeros or not. If ± 1 are not among zeros of the polynomial, the quotient is a direct sum of the algebras $\left(\mathbb{C}[u]/u^{L_j}\mathbb{C}[u]\right) \otimes \mathfrak{sl}_2$ for positive integers L_j. In the case when the polynomial is $(t \pm 1)^L$, we found by computer experiments that the derived algebras of the quotient algebras thus arising are identical to a series of nilpotent Lie algebra studied by Santharoubane [24]. Santharoubane obtained such a series in a project of reducing the classification of nilpotent Lie algebras to that of certain ideals in the nilpotent part of Kac–Moody–Lie algebras. The notion of roots in nilpotent Lie algebras are used there to establish the connection with Kac–Moody–Lie algebras.

Now we outline the contents of this paper. We start by recalling in section 2 some known facts about the Onsager algebra needed for our later discussion. In section 3, we establish a basic ideal-structure theorem of the Onsager algebra (see theorem 2), which associates each closed ideal with a reciprocal polynomial (in one variable t). Using this result, the structure of quotients of the Onsager algebra by closed ideals is reduced to those by ideals which are generated by powers of elementary reciprocal polynomials, $(t - a)(t - a^{-1})$ for non-zero complex numbers a. In sections 4 and 5, we derive the structure of those quotient algebras for $a \neq \pm 1$ and $a = \pm 1$, respectively. Furthermore, we study the completion of the Onsager algebra through these quotient algebras, which provides a different realization of the Onsager algebra in a completion of the nilpotent part of $A_1^{(1)}$ in the principal realization. The relation with solvable, nilpotent algebras is also discussed in section 5 for the case $a = 1$. In section 6, we derive the classification of finite-dimensional irreducible representations of the Onsager algebra, a result known in [9, 23], from our ideal theoretical point of view, and discuss its physical application to the spectra of the superintegrable chiral Potts Hamiltonian. Finally, we conclude in section 7 with some remarks.

Notation

To present our work, we introduce some notation. In this paper, we shall use the following conventions:

- $\mathfrak{sl}_2$ is the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ with the standard generators e, f, h,

$$[e, f] = h \qquad [h, e] = 2e \quad [e, f] = -2f.$$

- $\theta : \mathfrak{sl}_2 \longrightarrow \mathfrak{sl}_2$, the (Lie) involution defined by $\theta(e) = f, \theta(f) = e, \theta(h) = -h$.
- $L(\mathfrak{sl}_2) = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{sl}_2$, the loop algebra of $\mathfrak{sl}_2$ with the Lie-bracket

$$[p(t)x, q(t)y] = p(t)q(t)[x, y] \qquad \text{for} \quad p(t), q(t) \in \mathbb{C}[t, t^{-1}] \quad x, y \in \mathfrak{sl}_2.$$

- $\hat{\theta} : L(\mathfrak{sl}_2) \longrightarrow L(\mathfrak{sl}_2)$, the involution defined by $\hat{\theta}(p(t) \otimes x) = p(t^{-1}) \otimes \theta(x)$.
- $\mathfrak{sl}_2[[u]] = \mathbb{C}[[u]] \otimes \mathfrak{sl}_2$, the Lie algebra of formal series in u with coefficients in $\mathfrak{sl}_2$.
- For an (Lie) ideal $\mathfrak{I}$ of a (non-trivial) Lie algebra $\mathfrak{L}$ (over $\mathbb{C}$), we shall denote

$$Z(\mathfrak{I}) := \{x \in \mathfrak{L} \mid [x, \mathfrak{L}] \subset \mathfrak{I}\}$$

which is an ideal of $\mathfrak{L}$ such that $Z(\mathfrak{I})/\mathfrak{I}$ is the centre of the quotient algebra $\mathfrak{L}/\mathfrak{I}$. A (non-trivial) ideal $\mathfrak{I}$ is called a closed ideal if $Z(\mathfrak{I}) = \mathfrak{I}$, equivalently, $\mathfrak{L}/\mathfrak{I}$ is a Lie algebra with trivial centre.

2. The Onsager algebra

Let A_0 and A_1 be elements of a Lie algebra and denote

$$G_1 = \tfrac{1}{4}[A_1, A_0].$$

An infinite sequence of elements A_m, G_m ($m \in \mathbb{Z}$) is defined by the relations

$$A_{m-1} - A_{m+1} = \tfrac{1}{2}[A_m, G_1] \qquad G_m = \tfrac{1}{4}[A_m, A_0].$$

Theorem 1. *The following conditions are equivalent.*

(I) The elements A_0 and A_1 satisfy the Dolan–Grady (DG) condition [11]:

$$[A_1, [A_1, [A_1, A_0]]] = 16[A_1, A_0] \qquad [A_0, [A_0, [A_0, A_1]]] = 16[A_0, A_1].$$

(II) The infinite sequence of elements A_m, G_m ($m \in \mathbb{Z}$) satisfy the relation (1).

Proof. The proof of (I) $\Longrightarrow$ (II) can be found in [10, 23], which will be omitted here. For (II) $\Longrightarrow$ (I), it follows from the relations in (1a) for $(m, l) = (0, \pm 1), (1, 0)$ and (1b) for $(m, l) = (0, 1), (1, 1)$. □

By the above theorem, we introduce the following definition of the Onsager algebra [23].

Definition. *The Onsager algebra, denoted by $\mathfrak{OA}$, is the universal Lie algebra generated by two elements A_0 and A_1 with the DG condition. Equivalently, $\mathfrak{OA}$ is identified with the fixed Lie-subalgebra of $L(\mathfrak{sl}_2)$ by the involution $\hat{\theta}$. The elements A_m, G_m of $\mathfrak{OA}$ have the following expressions in $L(\mathfrak{sl}_2)$:*

$$A_m = 2t^m e + 2t^{-m} f \qquad G_m = (t^m - t^{-m})h \qquad \textit{for} \quad m \in \mathbb{Z}.$$

We shall always make the above identification of A_m and G_m as elements in $L(\mathfrak{sl}_2)$ in what follows. For an element X of $L(\mathfrak{sl}_2)$, the criterion of X in $\mathfrak{OA}$ is given by

$$X \in \mathfrak{OA} \quad \Longleftrightarrow \quad X = p(t)e + p(t^{-1})f + q(t)h \qquad \text{with} \quad q(t) = -q(t^{-1})$$

where $p(t), q(t) \in \mathbb{C}[t, t^{-1}]$. Note that a polynomial $q(t)$ with the above property can be written in the form

$$q(t) = q_+(t) - q_+(t^{-1}) \qquad \text{with} \quad q_+(t) \in \mathbb{C}[t].$$

Then the following equalities hold:

$$\begin{aligned} &[p(t)e + p(t^{-1})f + q(t)h, e + f] = 2q(t)e + 2q(t^{-1})f + (p(t) - p(t^{-1}))h \\ &[p(t)e + p(t^{-1})f + q(t)h, te + t^{-1}f] = 2tq(t)e - 2t^{-1}q(t)f + (t^{-1}p(t) - tp(t^{-1}))h. \end{aligned} \tag{2}$$

It is easy to see that the universal enveloping algebra of $\mathfrak{O}\mathfrak{A}$ is the fixed subalgebra of the universal enveloping algebra of $L(\mathfrak{sl}_2)$ by $\hat{\theta}$, hence with the inherited co-multiplication structure:

$$A_m \mapsto A_m \otimes 1 + 1 \otimes A_m \qquad G_m \mapsto G_m \otimes 1 + 1 \otimes G_m.$$

In $\mathfrak{O}\mathfrak{A}$, there are two involutions ι, σ defined by

$$\begin{aligned} &\iota : p(t)x \mapsto p(t^{-1})x \qquad \text{i.e.} \quad \iota(A_m) = A_{-m} \qquad \iota(G_m) = G_{-m} \\ &\sigma : p(t)x \mapsto p(-t)x \qquad \text{i.e.} \quad \sigma(A_m) = (-1)^m A_m \qquad \sigma(G_m) = (-1)^m G_m \end{aligned}$$

which we will use later in the paper.

3. Ideal structure of the Onsager algebra

In this section, we are going to determine the structure of closed ideals of $\mathfrak{O}\mathfrak{A}$. Let $P(t)$ be a non-trivial monic polynomial in $\mathbb{C}[t]$. We call $P(t)$ a reciprocal polynomial if $P(t) = \pm t^d P(t^{-1})$, where d is the degree of $P(t)$. Then one has, $P(t)\mathbb{C}[t, t^{-1}] = P(t^{-1})\mathbb{C}[t, t^{-1}]$. It is easy to see that the zeros of $P(t)$ not equal to ± 1 occur in reciprocal pairs. In fact, $P(t)$ is a product of the following elementary reciprocal polynomials $U_a(t)$ for $a \in \mathbb{C}^*$, where $U_a(t)$ is defined by

$$U_a(t) := \begin{cases} t^2 - (a + a^{-1})t + 1 & \text{if} \quad a^2 \neq 1 \\ t - a & \text{if} \quad a = \pm 1. \end{cases} \tag{3}$$

Note that $U_a(t) = U_{a^{-1}}(t)$. Let $P(t)$ be a reciprocal polynomial. We call an element X of $\mathfrak{O}\mathfrak{A}$ divisible by $P(t)$, denoted by $P(t)|X$, if $X = p(t)e + p(t^{-1})f + q(t)h$ with $p(t), q(t) \in P(t)\mathbb{C}[t, t^{-1}]$. Denote

$$\mathfrak{I}_{P(t)} := \{X \in \mathfrak{O}\mathfrak{A} \mid P(t)|X\}.$$

Then $\mathfrak{I}_{P(t)}$ is an ideal in $\mathfrak{O}\mathfrak{A}$ invariant under the involution ι. For two reciprocal polynomials $P(t)$ and $Q(t)$, one has the relation

$$\mathfrak{I}_{P(t)} \bigcap \mathfrak{I}_{Q(t)} = \mathfrak{I}_{\text{lcm}(P(t), Q(t))}.$$

In particular, $\mathfrak{I}_{P(t)} \subseteq \mathfrak{I}_{Q(t)}$ if $Q(t)|P(t)$, hence there is the canonical projection

$$\mathfrak{O}\mathfrak{A}/\mathfrak{I}_{P(t)} \longrightarrow \mathfrak{O}\mathfrak{A}/\mathfrak{I}_{Q(t)}.$$

The following lemma will be useful for later purposes.

Lemma 1. *Let $P_j(t) 1 \leqslant j \leqslant J$ be pairwise relatively prime reciprocal polynomials and $P(t) := \prod_{j=1}^{J} P_j(t)$. Then the canonical projections give rise to a Lie-isomorphism:*

$$\mathfrak{O}\mathfrak{A}/\mathfrak{I}_{P(t)} \xrightarrow{\sim} \prod_{j=1}^{J} \mathfrak{O}\mathfrak{A}/\mathfrak{I}_{P_j(t)}.$$

Proof. The injective part is obvious, so we only need to show the surjectivity of the above map. For $X_j = p_j(t)e + p_j(t^{-1})f + q_j(t)h \in \mathfrak{OA}$, $1 \leqslant j \leqslant J$, let N be a positive integer such that $t^N p_j(t), t^N q_j(t)$ are all polynomials in t. By the Chinese remainder theorem, there exist polynomials $\tilde{p}(t), \tilde{q}(t) \in \mathbb{C}[t]$ such that the following relations hold in $\mathbb{C}[t]$:

$$\tilde{p}(t) \equiv t^N p_j(t) \qquad \tilde{q}(t) \equiv t^N q_j(t) \pmod{P_j(t)} \qquad \text{for all } j.$$

Define the element X of $\mathfrak{OA}$ by

$$X := p(t)e + p(t^{-1})f + q(t)h$$

where

$$p(t) = \frac{\tilde{p}(t)}{t^N} \qquad q(t) = \frac{1}{2}\left(\frac{\tilde{q}(t)}{t^N} - t^N \tilde{q}(t^{-1})\right) \in \mathbb{C}[t, t^{-1}].$$

By $q_j(t) + q_j(t^{-1}) = 0$, one can easily see that both $q(t) - q_j(t)$, $p(t) - p_j(t)$ are divisible by $P_j(t)$ for all j. Hence $X \equiv X_j \pmod{\mathfrak{I}_{P_j(t)}}$ for all j. □

Lemma 2. *Let $P(t)$ be a reciprocal polynomial and write*

$$P(t) = (t-1)^L (t+1)^K P^*(t) \qquad L, K \geqslant 0 \quad P^*(\pm 1) \neq 0.$$

Denote $\widetilde{P}(t) := (t-1)^{2[L/2]}(t+1)^{2[K/2]} P^*(t)$, here $[r]$ stands for the integral part of a rational number r. Then

$$Z(\mathfrak{I}_{P(t)}) = \{p(t)e + p(t^{-1})f + q(t)h \in \mathfrak{OA} \mid \widetilde{P}(t)|p(t), P(t)|q(t)\}.$$

As a consequence, $\mathfrak{I}_{P(t)}$ is closed if and only if the zero-multiplicities of $P(t)$ at $t = \pm 1$ are even.

Proof. For $X = p(t)e + p(t^{-1})f + q(t)h \in \mathfrak{OA}$, the criterion of X in $Z(\mathfrak{I}_{P(t)})$ is given by the relations

$$P(t)|[X, e+f], [X, te + t^{-1}f]$$

which by (2) is the same as

$$P(t)|q(t), (t - t^{-1})p(t), p(t) - p(t^{-1}).$$

Let $p(t)$ be an element of $\mathbb{C}[t, t^{-1}]$ which satisfies the above conditions. Then $P^*(t)|p(t)$. In order to have the conclusion of the lemma, we may assume either L or K to be greater than 0, say $L > 0$. One has

$$(t-1)^{L-1}|p(t), (t-1)^L|(p(t) - p(t^{-1})). \tag{4}$$

Write $p(t) = (t-1)^{L-1}h(t)$. We have

$$p(t) - p(t^{-1}) = (t-1)^{L-1}\left(h(t) - (-t)^{-L+1}h(t^{-1})\right).$$

Then $(t-1)|(h(t) - (-t)^{-L+1}h(t^{-1}))$, i.e. $h(1)(1 - (-1)^{-L+1}) = 0$, which is equivalent to the following relation:

$$(t-1)^L|p(t) \qquad \text{for even } L$$
$$(t-1)^{L-1}|p(t) \qquad \text{for odd } L.$$

Therefore, the relation (4) is the same as $(t-1)^{2[L/2]}|p(t)$. The same argument also applies to the case $(t+1)^K$ for $K > 0$. Hence we obtain the following result. □

Lemma 3. *Let I be an ideal in $\mathfrak{O}\mathfrak{A}$ and $r(t)$ be an element of $\mathbb{C}[t, t^{-1}]$.*

(i) If $r(t)e + r(t^{-1})f$ is an element in I, then $(p(t) - p(t^{-1}))h \in I$ for $p(t) \in r(t)\mathbb{C}[t, t^{-1}]$.
(ii) For a closed ideal I and an integer l, one has

$$(t^j r(t) - t^{-j} r(t^{-1}))h \in I \quad (j = 0, -1) \quad \Longrightarrow \quad p(t^{\pm 1})e + p(t^{\mp 1})f \in I$$
$$\text{for} \quad p(t) \in r(t)\mathbb{C}[t, t^{-1}]$$

$$\begin{aligned} r(t)e + r(t^{-1})f \in I &\Longleftrightarrow r(t^{-1})e + r(t)f \in I \\ &\Longleftrightarrow t^l r(t)e + t^{-l} r(t^{-1})f \in I \\ &\Longleftrightarrow (t^j r(t) - t^{-j} r(t^{-1}))h \in I \qquad (j = 0, -1). \end{aligned}$$

Proof. It is easy to see that the equivalent relations in the second part of (ii) follows from the other ones of the lemma. We may also assume the $p(t)$ in the statement with the form $p(t) = t^m r(t)$ for $m \in \mathbb{Z}$. Write $r(t) = \sum_k a_k t^k$, where $a_k \in \mathbb{C}$, $a_k = 0$ for $|k| \gg 0$. One has

$$\begin{aligned} 2r(t)e + 2r(t^{-1})f &= \sum_k a_k A_k \\ (r(t) - r(t^{-1}))h &= \sum_k a_k G_k \\ (t^{-1} r(t) - t r(t^{-1}))h &= \sum_k a_k G_{k-1}. \end{aligned}$$

By the relation $(1a)$ one obtains (i). If I is a closed ideal and $\sum_m a_k G_{k+j} \in I$ for $j = 0, -1$ by $(1a)$ we have

$$4 \sum_m a_k G_k = \left[\sum_m a_k A_k, A_0 \right] \qquad 4 \sum_m a_k G_{k-1} = \left[\sum_m a_k A_k, A_1 \right]$$

which implies $r(t)e + r(t^{-1})f \in I$. Using (i), one has $(t^{m+j} r(t) - t^{-m-j} r(t^{-1}))h \in I$ for $j = 0, \pm 1$. With the same argument, we can also show $p(t^{\pm 1})e + p(t^{\mp 1})f \in I$. □

Theorem 2. *Let I be an ideal in $\mathfrak{O}\mathfrak{A}$. Then I is closed if and only if $I = \mathfrak{I}_{P(t)}$ for a reciprocal polynomial $P(t)$ whose zeros at $t = \pm 1$ are of even multiplicity.*

Proof. The 'if' part follows from lemma 2. Let I be a closed ideal. Denote

$$\bar{I} := \{r(t) \in \mathbb{C}[t, t^{-1}] \mid r(t)e + r(t^{-1})f \in I\}.$$

By lemma 3 (ii), $\bar{I}$ is an ideal in $\mathbb{C}[t, t^{-1}]$ invariant under the involution $r(t) \mapsto r(t^{-1})$. Let $P(t)$ be the monic polynomial which generates the ideal $\bar{I} \cap \mathbb{C}[t]$ of the polynomial ring $\mathbb{C}[t]$. Then $P(t)$ is a reciprocal polynomial and $\bar{I} = P(t)\mathbb{C}[t, t^{-1}]$. By (2), an element $q(t)h$ of $\mathfrak{O}\mathfrak{A}$ is divisible by $P(t)$ if and only if it belongs to I. Hence I contains the ideal $\mathfrak{I}_{P(t)}$. We are going to show $I = \mathfrak{I}_{P(t)}$, by which and using lemma 2, the result follows immediately. Otherwise, there exists an element X of $I \setminus \mathfrak{I}_{P(t)}$ and write

$$X = p(t)e + p(t^{-1})e + q(t)h \qquad q(t) = q_+(t) - q_+(t^{-1})$$

where $p(t) \in \mathbb{C}[t, t^{-1}]$, $q_+(t) \in \mathbb{C}[t]$. Note that $q(t)$ is not divisible by $P(t)$, neither for $p(t)$ by the first equality in (2). We may assume the polynomial $q_+(t)$ with the degree being less than that of $P(t)$ and $q_+(0) = 0$. Let $\tilde{X}$ be such an element X with the degree of $q_+(t)$ being the maximal one. By (2), we have the following expressions of elements in I:

$$[\tilde{X}, e + f] = 2q(t)e + 2q(t^{-1})f + k(t)h$$

$$[[\tilde{X}, e + f], te + t^{-1}f] = 2tk(t)e - 2t^{-1}k(t)f + 2(t^{-1}q(t) - tq(t^{-1})h.$$

Note that both $q(t)$ and $k(t)$ are not divisible by $P(t)$. One can write $t^{-1}q(t) - tq(t^{-1}) = \widetilde{q}(t) - \widetilde{q}(t^{-1})$, where $\widetilde{q}(t) := tq_+(t) + t^{-1}q_+(t) \in \mathbb{C}[t]$. The degree of the polynomial $\widetilde{q}(t)$ is greater than that of $q_+(t)$, so is $\widetilde{q}(t) - \widetilde{q}(0)$. By the definition of $\tilde{X}$, $t^{-1}q(t) - tq(t^{-1})$ is divisible by $P(t)$, which implies $P(t)|tk(t)$. By $P(0) \neq 0$, $k(t)$ is divisible by $P(t)$, which contradicts our previous statement for $k(t)$. □

Remark. The ideal $\mathfrak{I}_{P(t)}$ in the above theorem is characterized as the minimal closed ideal of $\mathfrak{OA}$ containing $P(t)e + P(t^{-1})f$. We shall call $P(t)$ the generating polynomial of the closed ideal $\mathfrak{I}_{P(t)}$.

Remark. For affine Lie algebras classification of ideals is known (cf theorem 4 of [18] and lemma 8.6 of [13]).

An important class of closed ideals arises from representations of $\mathfrak{OA}$, by which we shall always mean finite-dimensional Lie-algebra representations in this paper. The kernel of a representation $\rho : \mathfrak{OA} \longrightarrow \mathfrak{gl}(V)$ is always an ideal of $\mathfrak{OA}$, which will be denoted by $\mathrm{Ker}(\rho)$. As the DG condition is unchanged by adding constants on $\rho(A_0)$ and $\rho(A_1)$, one may assume that the representation takes values in $\mathfrak{sl}(V)$. For an irreducible representation ρ of $\mathfrak{OA}$ in $\mathfrak{sl}(V)$, by Schur's lemma $\mathrm{Ker}(\rho)$ is a closed ideal, and hence invariant under the involution ι by theorem 2. A representation ρ of $\mathfrak{OA}$ on a vector space V is called Hermitian if both $\rho(A_0)$ and $\rho(A_1)$ are Hermitian operators on V, which is equivalent to the Hermitian property of $\rho(A_j)$ for all j. Note that in this situation, the operators $\sqrt{-1}\rho(A_0)$ and $\sqrt{-1}\rho(A_0)$ give rise to a representation of $\mathfrak{OA}$ into $\mathfrak{su}(V)$. Hence every Hermitian representation of $\mathfrak{OA}$ is completely reducible. For an irreducible Hermitian representation ρ of $\mathfrak{OA}$ in $\mathfrak{sl}(V)$, $\mathrm{Ker}(\rho)$ is invariant under complex conjugation, equivalently the generating polynomial $P(t)$ of $\mathrm{Ker}(\rho)$ has real coefficients.

By theorem 2 and lemma 1, the understanding of the complete structure of quotient algebras of $\mathfrak{OA}$ by closed ideals I is reduced to the case $I = \mathfrak{I}_{U_a^L(t)}$ for $a \in \mathbb{C}^*$, $L \in \mathbb{Z}_{\geqslant 0}$, where $U_a(t)$ is the elementary reciprocal polynomial (3). For $a \in \mathbb{C}^*$, we are going to define a completion $\widehat{\mathfrak{OA}}_a$ of $\mathfrak{OA}$ as follows. Set $\mathfrak{OA}_{a,L} := \mathfrak{OA}/\mathfrak{I}_{U_a^L(t)}$ and we have the canonical projections

$$\pi_{a,L}: \quad \mathfrak{OA} \longrightarrow \mathfrak{OA}_{a,L}$$

$$\pi_{a,KL}: \quad \mathfrak{OA}_{a,L} \longrightarrow \mathfrak{OA}_{a,K} \qquad L \geqslant K \geqslant 0.$$

The projective system of Lie algebras $(\mathfrak{OA}_{a,L}, \pi_{a,KL})_{L,K \in \mathbb{Z}_{\geqslant 0}}$ gives rise to a limit, which is a Lie algebra denoted by

$$\widehat{\mathfrak{OA}}_a := \lim_{\leftarrow} \mathfrak{OA}_{a,L}.$$

For $L \geqslant 0$, there is a canonical morphism $\psi_{a,L} : \widehat{\mathfrak{OA}}_a \longrightarrow \mathfrak{OA}_{a,L}$. We denote its kernel by $\widehat{\mathfrak{OA}}_a^L := \mathrm{Ker}(\psi_{a,L})$. The ideals $\widehat{\mathfrak{OA}}_a^L$ form a filtration of $\widehat{\mathfrak{OA}}_a$,

$$\widehat{\mathfrak{OA}}_a = \widehat{\mathfrak{OA}}_a^0 \supset \widehat{\mathfrak{OA}}_a^1 \supset \cdots \supset \widehat{\mathfrak{OA}}_a^L \supset \cdots \qquad \widehat{\mathfrak{OA}}_a / \widehat{\mathfrak{OA}}_a^L \simeq \mathfrak{OA}_{a,L}.$$

The family of morphisms $\{\psi_{a,L}\}$ gives rise to a morphism from $\mathfrak{OA}$ into $\widehat{\mathfrak{OA}}_a$

$$\pi_a : \mathfrak{OA} \longrightarrow \widehat{\mathfrak{OA}}_a$$

with $\psi_{a,L}\pi_a = \pi_{a,L}$. In the next two sections, we are going to determine the structure of $\mathfrak{OA}_{a,L}$ and $\widehat{\mathfrak{OA}}_a$. For $L = 1$, $\mathfrak{OA}_{a,1}$ can be realized in $\mathfrak{sl}_2$ through the evaluation morphism of $\mathfrak{OA}$

$$ev_a : \mathfrak{OA} \longrightarrow \mathfrak{sl}_2 \qquad p(t)x \mapsto p(a)x.$$

In fact, we have the following result.

Lemma 4. $\mathrm{Ker}(ev_a) = \mathfrak{I}_{U_a(t)}$ *and the map* ev_a *induces the isomorphism:*

$$\mathfrak{OA}_{a,1} \simeq \begin{cases} \mathbb{C}(e+f) & \text{if} \quad a = \pm 1 \\ \mathfrak{sl}_2 & \text{otherwise.} \end{cases}$$

Proof. By $U_a(a^{\pm 1}) = 0$, $\mathrm{Ker}(ev_a) \supseteq \mathfrak{I}_{U_a(t)}$. Let X be an element of $\mathrm{Ker}(ev_a)$ with the expression $X = p(t)e + p(t^{-1})f + q(t)h$. Then $p(a^{\pm 1}) = q(a) = 0$. By $q(t^{-1}) = -q(t)$, $p(t)$ and $q(t)$ are divisible by $U_a(t)$, therefore $X \in \mathfrak{I}_{U_a(t)}$. When $a = \pm 1$, it is easy to see that $\mathrm{Im}(ev_a)$ is the one-dimensional space generated by $e + f$. For $a \neq \pm 1$, $\mathrm{Im}(ev_a)$ is a Lie subalgebra of $\mathfrak{sl}_2$ containing $e + f, ae + a^{-1}f$. This implies $e, f \in \mathrm{Im}(ev_a)$, hence $\mathrm{Im}(ev_a) = \mathfrak{sl}_2$. In fact, the basis of $\mathfrak{sl}_2$ has the following expression in terms of elements of $\mathrm{Im}(ev_a)$, $\overline{A}_m := ev_a(A_m)$:

$$e = \frac{\overline{A}_1 - a^{-1}\overline{A}_0}{2(a - a^{-1})} \qquad f = \frac{\overline{A}_1 - a\overline{A}_0}{-2(a - a^{-1})} \qquad h = \frac{\overline{G}_1}{a - a^{-1}}$$

or

$$e = \frac{a\overline{A}_m - \overline{A}_{m-1}}{2a^m(a - a^{-1})} \qquad f = \frac{a^{-1}\overline{A}_m - \overline{A}_{m-1}}{-2a^{-m}(a - a^{-1})} \qquad h = \frac{\overline{G}_m}{a^m - a^{-m}}. \qquad \square$$

4. Structure of the quotient by ideal generated by $U_a(t)^L$, $a \neq \pm 1$

In the following discussion, we use the following notation for the shifted factorial:

$$x^{(0)} = 1 \qquad x^{(n)} = x(x-1)\cdots(x-n+1) \qquad n \in \mathbb{Z}_{>0}.$$

Then

$$\binom{x}{n} = \frac{x^{(n)}}{k!}.$$

For $a \in \mathbb{C}^*$, we define an injective (Lie) morphism from $\mathfrak{OA}$ into $\mathfrak{sl}_2[[u]]$ by the Taylor series expansion of a function around $t = a$ with the variable $u = t - a$,

$$s_a : \mathfrak{OA} \longrightarrow \mathfrak{sl}_2[[u]] \qquad p(t)x \mapsto \sum_{j \geqslant 0} \frac{\mathrm{d}^j p}{\mathrm{d}t^j}(a)\frac{u^j}{j!}x \tag{5}$$

where $p(x) \in \mathbb{C}[t, t^{-1}]$, $x \in \mathfrak{sl}_2$. For a positive integer L, $\mathfrak{sl}_2[[u]]/u^L\mathfrak{sl}_2[[u]]$ is isomorphic to the Lie algebra $(\mathbb{C}[u]/u^L\mathbb{C}[u]) \otimes \mathfrak{sl}_2$. By composing with the natural projection of $\mathfrak{sl}_2[[u]]$ to $\mathfrak{sl}_2[[u]]/u^L\mathfrak{sl}_2[[u]]$, s_a gives rise to the morphism

$$s_{a,L} : \mathfrak{OA} \longrightarrow \mathfrak{sl}_2[[u]]/u^L\mathfrak{sl}_2[[u]].$$

Note that $s_{a,1}$ is equivalent to the evaluation morphism ev_a. We are going to extend the result of lemma 4 to an arbitrary positive integer L. First we need the following lemma.

Lemma 5. *Let $f(t)$ be an analytic function on $\mathbb{C}^*$. Denote $\check{f}(t) = f(t^{-1})$. For $a \in \mathbb{C}^*$, the zero multiplicities of $f(t)$ at a^{-1} and $\check{f}(t)$ at a are the same.*

Proposition 1. *For $L \in \mathbb{Z}_{>0}$, $\mathrm{Ker}(s_{a,L}) = \mathfrak{I}_{U_a^L}$. The morphism $s_{a,L}$ is surjective if and only if $a \neq \pm 1$.*

Proof. As before we denote $\check{f}(t) = f(t^{-1})$ for $f(t) \in \mathbb{C}[t, t^{-1}]$. An element of $\mathfrak{O}\mathfrak{A}$ is expressed by $p(t)e + \check{p}(t)f + q(t)h$ with $p(t), q(t) \in \mathbb{C}[t, t^{-1}]$ and $q(t) = -\check{q}(t)$. By lemma 5, we have

$$\begin{aligned} p(t)e + \check{p}(t)f + q(t)h \in \mathfrak{I}_{U_a^L(t)} &\iff p(t), q(t) \in (t - a^{\pm 1})^L \mathbb{C}[[t - a^{\pm 1}]] \\ &\iff p(t), \check{p}(t), q(t) \in (t-a)^L \mathbb{C}[[t-a]] \\ &\iff p(t)e + \check{p}(t)f + q(t)h \in \mathrm{Ker}(s_{a,L}). \end{aligned}$$

Therefore, $\mathfrak{I}_{U_a^L} = \mathrm{Ker}(s_{a,L})$. For $a \neq \pm 1$, $U_a^L(t)$ is a polynomial of degree $2L$. One has the relation

$$\begin{pmatrix} s_{a,L}(A_0) \\ s_{a,L}(A_1) \\ \vdots \\ \vdots \\ s_{a,L}(A_{2L-2}) \\ s_{a,L}(A_{2L-1}) \end{pmatrix} = 2(C_+, C_-) \begin{pmatrix} e \\ \vdots \\ \dfrac{u^{L-1}}{(L-1)!} e \\ f \\ \vdots \\ \dfrac{u^{L-1}}{(L-1)!} f \end{pmatrix}.$$

where $C_\pm$ are the $2L \times L$-matrices defined by

$$C_\pm = \begin{pmatrix} a_{0,0} & \cdots & a_{0,L-1} \\ a_{\pm 1,0} & \cdots & a_{\pm 1,L-1} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ a_{\pm(2L-2),0} & \cdots & a_{\pm(2L-2),L-1} \\ a_{\pm(2L-1),0} & \cdots & a_{\pm(2L-1),L-1} \end{pmatrix} \qquad a_{m,j} := m^{(j)} a^{m-j}.$$

Claim. *The square matrix (C_+, C_-) is invertible.*

Otherwise there exists a non-trivial linear relation $\sum_{m=0}^{2L-1} \alpha_m a_{\pm m,j} = 0$ $(j = 0, \ldots, L-1)$ for $\alpha_m \in \mathbb{C}$ not all zeros. Then $\sum_{m=0}^{2L-1} \alpha_m t^m$ is a non-trivial polynomial, denoted by $Q(t)$. By lemma 5, the zero-multiplicities of the polynomial $Q(t)$ at $t = a^{\pm 1}$ are both at least L, which contradicts the fact that the degree of $Q(t) < 2L$. By inverting the square matrix (C_+, C_-), the elements $(u^j/j!)e$ and $(u^j/j!)f$ for $j = 0, \ldots, L-1$, are in the Lie subalgebra $\mathrm{Im}(s_{a,L})$ of $\mathfrak{sl}_2[[u]]/u^L\mathfrak{sl}_2[[u]]$. Hence it follows immediately that the surjectivity of $s_{a,L}$ for $a \neq \pm 1$. For $a = \pm 1$, we consider the commutative diagram

$$\begin{array}{ccc} \mathfrak{O}\mathfrak{A} & \xrightarrow{s_{a,L}} & \mathfrak{sl}_2[[u]]/u^L\mathfrak{sl}_2[[u]] \\ \| & & \downarrow \\ \mathfrak{O}\mathfrak{A} & \xrightarrow{s_{a,1}} & \mathfrak{sl}_2 \end{array}$$

where the right vertical morphism is the canonical projection. By lemma 4, the image of $s_{\pm 1,1}$ is an one-dimensional subspace of $\mathfrak{sl}_2$ and it implies the non-surjectivity of $s_{\pm 1,L}$. □

As a corollary of proposition 1, for $a \neq \pm 1$ the morphism $s_{a,L}$ induces an isomorphism,

$$\mathfrak{OA}_{a,L} \simeq \mathfrak{sl}_2[[u]]/u^L\mathfrak{sl}_2[[u]] \qquad L \in \mathbb{Z}_{>0}.$$

By the definition of the formal algebra $\widehat{\mathfrak{OA}}_a$, one obtains the following result.

Theorem 3. *For $a \in \mathbb{C}^*$ and $a \neq \pm 1$, the morphism s_a of (5) gives rise to the following isomorphisms:*

$$\widehat{\mathfrak{OA}}_a \simeq \mathfrak{sl}_2[[u]] \qquad \widehat{\mathfrak{OA}}_a^L \simeq u^L\mathfrak{sl}_2[[u]]$$

and

$$\mathfrak{OA}_{a,L} \simeq (\mathbb{C}[u]/u^L\mathbb{C}[u]) \otimes \mathfrak{sl}_2.$$

Remark. For $a = \sqrt{-1}$, one has $U_a(t) = t^2 + 1$. The above structure of $\mathfrak{OA}_{\sqrt{-1},L}$ for $L = 2$ appeared in [23].

Through the morphism π_a, one can regard $\mathfrak{OA}$ as a subalgebra of $\mathfrak{sl}_2[[u]]$, which is identified with $\widehat{\mathfrak{OA}}_a$ by theorem 3. Then the expressions of A_m and G_m in $\mathfrak{sl}_2[[u]]$ are given by

$$A_m = 2\sum_{j\geqslant 0}\left(m^{(j)}a^{m-j}\frac{u^j e}{j!} + (-m)^{(j)}a^{-m-j}\frac{u^j f}{j!}\right)$$

$$G_m = \sum_{j\geqslant 0}\left(m^{(j)}a^{m-j} - (-m)^{(j)}a^{-m-j}\right)\frac{u^j h}{j!}.$$

With the identification $\mathfrak{OA}_{a,L}$ with $\mathfrak{sl}_2[[u]]/u^L\mathfrak{sl}_2[[u]]$, one has the expression of elements of the Onsager algebra,

$$A_m = 2\sum_{j=0}^{L-1}(m^{(j)}a^{m-j}e_j + (-m)^{(j)}a^{-m-j}f_j)$$

$$G_m = \sum_{j=0}^{L-1}(m^{(j)}a^{m-j} - (-m)^{(j)}a^{-m-j})h_j$$

where e_j, f_j and h_j are the elements in $\mathfrak{sl}_2[[u]]/u^L\mathfrak{sl}_2[[u]]$ represented by $(u^j/j!)e$, $(u^j/j!)f$, $(u^j/j!)h$, respectively. In fact, one can start with the above relations. Using Onsager's relation (1) of A_m and G_m, one obtains the relations of e_j, f_j and h_j by the following formula of shift factorials:

$$(x+y)^{(m)} = \sum_{k=0}^{m}\binom{m}{k}x^{(k)}y^{(m-k)}.$$

5. Structure of the quotient using an ideal generated by $(t \pm 1)^L$

In this section, we shall discuss the structure of $\mathfrak{OA}_{a,L}$ for $a = \pm 1$. As the involution σ of $\mathfrak{OA}$, which sends t to $-t$, induces an isomorphism

$$\mathfrak{OA}_{1,L} \simeq \mathfrak{OA}_{-1,L}$$

we only need to consider the case $a = 1$. For the simplicity of notation, we shall omit the index $a = 1$ in this section and denote the algebras as

$$\mathfrak{OA}_L = \mathfrak{OA}_{1,L} \qquad \widehat{\mathfrak{OA}} = \widehat{\mathfrak{OA}}_1 \qquad \widehat{\mathfrak{OA}}^L = \widehat{\mathfrak{OA}}_1^L$$

and the morphisms as

$$\pi_L : \mathfrak{OA} \longrightarrow \mathfrak{OA}_L \qquad \pi : \mathfrak{OA} \longrightarrow \widehat{\mathfrak{OA}} \qquad \psi_L : \widehat{\mathfrak{OA}} \longrightarrow \mathfrak{OA}_L .$$

For $X \in \mathfrak{OA}$, the element $\pi(X)$ in $\widehat{\mathfrak{OA}}$ will be denoted by X again later in the discussion.

Lemma 6. *There are unique elements* $\underline{X}_k, \underline{Y}_k$ $(0 \leqslant k < L)$ *in* $\mathfrak{OA}_L$ *such that*

$$\pi_L(A_m) = \sum_{k=0}^{L-1} \binom{m}{k} \underline{X}_k \qquad \pi_L(G_m) = \sum_{k=0}^{L-1} (-1)^k \binom{m}{k} \underline{Y}_k \qquad \text{for} \quad m \in \mathbb{Z}.$$

Furthermore, we have $\underline{Y}_k := \frac{1}{4}(-1)^k[\underline{X}_k, \underline{X}_0]$.

Proof. The above relations for $0 \leqslant m < L$ imply the uniqueness of $\underline{X}_k, \underline{Y}_k$. In fact, one can define $\underline{X}_k$ in terms of $\pi_L(A_m)$ $(0 \leqslant m < L)$ through these relations and set $\underline{Y}_k := \frac{1}{4}[\underline{X}_k, \underline{X}_0]$. By the definition of the ideal $\mathfrak{I}_{(t-1)^L}$, we have

$$\sum_{k=0}^{L-1} (-1)^k \binom{L}{k} \pi_L(A_{k+m}) = 0 \qquad m \in \mathbb{Z}.$$

By using $(1a)$, the results follow. □

The above elements $\underline{X}_k, \underline{Y}_k$ of $\mathfrak{OA}_L$ for $L > 0$ give rise to an infinite sequence of elements in $\widehat{\mathfrak{OA}}$, X_k, Y_k $(k \in \mathbb{Z}_{\geqslant 0})$ such that the following relations hold in $\widehat{\mathfrak{OA}}$:

$$A_m (= \pi(A_m)) = \sum_{k \geqslant 0} \binom{m}{k} X_k \qquad G_m (= \pi(G_m)) = \sum_{k \geqslant 0} (-1)^k \binom{m}{k} Y_k . \quad (6)$$

In $\mathfrak{OA}_L$, we have

$$\psi_L(X_k) = \begin{cases} \underline{X}_k & \text{for} \quad 0 \leqslant k < L \\ 0 & \text{otherwise.} \end{cases}$$

For simplicity of notation, the element $\psi_L(X)$ of $\mathfrak{OA}_L$ for $X \in \widehat{\mathfrak{OA}}$ will sometimes again be denoted by X in the later discussions if no confusion could arise.

Recall that the Stirling numbers of the first kind

$$s_k^n \qquad (n, k \in \mathbb{Z}_{\geqslant 0})$$

are integers such that the following relations hold for the shifted factorial $x^{(n)}$ and the kth power x^k:

$$x^{(n)} = \sum_{k \geqslant 0} x^k s_k^n .$$

Then one has

$$\binom{x}{n} = \sum_{k \geqslant 0} \frac{1}{k!} s_k^n x^k .$$

Substituting the relations (6) into the defining relation of the Onsager algebra (1) and comparing the coefficients of $m^a l^b$, we have

$$\sum_{n,k \geqslant 0} \frac{1}{n!k!} s_a^n s_b^k [X_n, X_k] = 4(-1)^b \binom{a+b}{b} \sum_{k \geqslant 0} \frac{(-1)^k}{k!} s_{a+b}^k Y_k$$

$$\sum_{n,k \geqslant 0} \frac{(-1)^n}{n!k!} s_a^n s_b^k [Y_n, X_k] = 2(1 - (-1)^a) \binom{a+b}{b} \sum_{k \geqslant 0} \frac{1}{k!} s_{a+b}^k X_k$$

$$[Y_n, Y_k] = 0.$$

The Stirling numbers of the second kind S_k^n $(n, k \in \mathbb{Z}_{\geqslant 0})$ satisfy the relation

$$x^n = \sum_{k\geqslant 0} x^{(k)} S_k^n.$$

(For a general discussion of Stirling numbers, see, e.g., [8, 21].) Note that $s_n^n = S_n^n = 1$ and $s_k^n = S_k^n = 0$ for $k > n$. As a direct consequence of the definition the matrices of infinite size $\left(s_k^n\right)_{n,k\geqslant 0}$ and $\left(S_k^n\right)_{n,k\geqslant 0}$ are inverse to each other

$$\sum_k s_k^a S_b^k = \delta_b^a. \tag{7}$$

For Stirling numbers we have the following identities:

Lemma 7.

$$\sum_k (-1)^k s_k^a S_b^k = (-1)^a \frac{a!}{b!}\binom{a-1}{b-1} \quad \text{[21] (p 44)}$$

$$\binom{j+k}{k} S_a^j = \sum_l \binom{a+l}{l} s_k^l S_{a+l}^{j+k} \quad \text{[22] (p 204).}$$

Note that the numbers defined by the first relation are known as the Lah numbers. Using (7) and the identities in lemma 7, we obtain

Proposition 2.

$$\begin{aligned}
[X_n, X_k] &= 4(-1)^n \sum_{a\geqslant 0} \binom{a+k-1}{a} Y_{a+n+k} \\
&= 4(-1)^n \left(Y_{n+k} + \sum_{a>0} \binom{a+k-1}{a} Y_{a+n+k} \right) \\
[Y_n, X_k] &= 2\left((-1)^n X_{n+k} - \sum_{a\geqslant 0} (-1)^a \binom{a+n-1}{a} X_{a+n+k} \right) \\
&= 2\left(\left((-1)^n - 1\right) X_{n+k} - \sum_{a>0} (-1)^a \binom{a+n-1}{a} X_{a+n+k} \right) \\
[Y_n, Y_k] &= 0.
\end{aligned} \tag{8}$$

With the infinite formal sum, $\widehat{\mathfrak{O}\mathfrak{A}}$ is a formal Lie algebra with generators X_k, Y_k; while $\widehat{\mathfrak{O}\mathfrak{A}}^L$ is the ideal generated by X_k, Y_k with $k \geqslant L$. Hence for $L \geqslant 0$, $\widehat{\mathfrak{O}\mathfrak{A}}^L/\widehat{\mathfrak{O}\mathfrak{A}}^{L+1}$ is Abelian and $\widehat{\mathfrak{O}\mathfrak{A}}/\widehat{\mathfrak{O}\mathfrak{A}}^L$ is a finite-dimensional solvable Lie algebra. The elements Y_k are not independent in $\widehat{\mathfrak{O}\mathfrak{A}}$. We are now going to describe the relations among Y_ks. Since the commutator is skew symmetric $[X_n, X_k] = -[X_k, X_n]$, from the first relation of (8) we have

$$\sum_{a\geqslant 0} \left(\binom{a+n-k-1}{a} + (-1)^n \binom{a+k-1}{a} \right) Y_{a+n} = 0 \qquad 0 \leqslant k \leqslant n. \tag{9}$$

Since one easily sees that $(9)_{n+1,k+1} = (9)_{n,k+1} - (9)_{n,k}$, and $(9)_{n,n} = (9)_{n,0}$, the relation (9) is reduced to the following one:

$$\left(1 + (-1)^n\right) Y_n + \sum_{a\geqslant 1} \binom{a+n-1}{a} Y_{a+n} = 0 \qquad \text{for} \quad n \geqslant 0. \tag{10}$$

In particular, we have

$$Y_0 = 0 \qquad 2Y_{2n} + \sum_{a\geqslant 1} \binom{a+2n-1}{2n-1} Y_{a+2n} = 0 \qquad \text{for} \quad n \geqslant 1. \tag{11}$$

In order to solve this constraint, we first consider their relations in $\mathfrak{O}\mathfrak{A}_L$. In this situation, we easily see that $Y_L = 0$ if L is odd. For simplicity, we assume at this moment that L is even. From (10), we see that Y_{2j} is a linear combination of Y_{2k+1}, $j \leqslant k \leqslant [\frac{1}{2}(L-1)]$ and write

$$Y_{2j} = \sum_{k=j}^{[\frac{1}{2}(L-1)]} \gamma_{jk} Y_{2k+1}. \tag{12}$$

After some calculation, we find that γ_{jk} has the following form:

$$\gamma_{jk} = \frac{(-1)^{k-j-1}}{2(k-j+1)} \binom{2k}{2j-1} \alpha_{k-j}. \tag{13}$$

Substituting (13) into (12), we have

$$\sum_{j=0}^{k} \frac{(-1)^j}{2j+2} \binom{2k+2}{2j+1} \alpha_j = 1 \qquad 0 \leqslant j \leqslant \left[\tfrac{1}{2}(L-1)\right]. \tag{14}$$

Note that this relation does not contain L. In order to solve (14) we employ the techniques from inversion relations (see [22], p 109).

Lemma 8. *If a sequence $\{a_k\}$ is expressed in terms of another sequence $\{b_k\}$ as*

$$a_{2n+1} = \sum_{k=0}^{n} \binom{2n+2}{2k+1} b_{2n+1-2k}$$

then we have the inversion formula

$$(2n+2)b_{2n+1} = \sum_{k=0}^{n} \binom{2n+2}{2k} d_{2k} a_{2n+1-2k}$$

where d_j are defined by the following expansion:

$$\frac{2x}{e^x - e^{-x}} = \sum_{j=0}^{\infty} \frac{d_{2j} x^{2j}}{(2n)!}.$$

The number d_{2j} is related to the Bernoulli number B_j by the relation

$$d_{2j} = (-1)^j (2^{2j} - 2) B_{2j} \qquad j \geqslant 1.$$

Recall that the Bernoulli numbers B_j are defined by the expansion

$$\frac{x}{e^x - 1} = \sum_{j=0}^{\infty} \frac{b_j}{j!} x^j \qquad \text{with} \quad B_j = (-1)^{j-1} b_{2j}.$$

Applying the inversion relation of lemma 8 to (14), one obtains

$$\alpha_j = (-1)^j \sum_{k=0}^{j} (-1)^k \binom{2j+2}{2k} (2^{2k} - 2) B_k. \tag{15}$$

On the other hand, we have the following relation for the Bernoulli numbers:

Lemma 9 (see [26]). *Denote* $v_j = 2(2^{2j} - 1)B_j$. *We have the following recurrence relation:*

$$v_k - \tfrac{1}{2}\binom{2k}{2}v_{k-1} + \tfrac{1}{2}\binom{2k}{4}v_{k-2} - \cdots + (-1)^{k-1}\tfrac{1}{2}\binom{2k}{2k-2}v_1 + (-1)^k k = 0.$$

Applying the above relation to the right-hand side of (15), we have

$$\alpha_j = 2(2^{2j+2} - 1)B_{j+1}.$$

Therefore, the following result follows immediately:

Proposition 3. *The following relations of* Y_k*s hold in* $\mathfrak{OA}_L$, *hence in* $\mathfrak{OA}$

$$Y_{2n} = \sum_{k\geqslant n}(-1)^{k-n+1}\frac{(4^{k-n+1} - 1)B_{k-n+1}}{k-n+1}\binom{2k}{2n-1}Y_{2k+1}. \tag{16}$$

Through the morphism (5) for $a = 1$, $\widehat{\mathfrak{OA}}$ is embedded in the formal algebra $\mathfrak{sl}_2[[u]]$, where u is the local coordinate of the t-plane near $t = 1$

$$u = t - 1.$$

Another local coordinate near $t = 1$ is given by

$$v = t^{-1} - 1$$

with the relation $u = -v/(1 + v)$, $v = -u/(1 + u)$. We have $\mathfrak{sl}_2[[u]] = \mathfrak{sl}_2[[v]]$. In the $\mathfrak{sl}_2$-formal algebra, the elements X_k and Y_k have the following symmetric expressions.

Lemma 10. *The generators* X_k, Y_k $(k \geqslant 0)$ *of* $\widehat{\mathfrak{OA}}$ *can be represented by*

$$X_k = 2u^k e + 2v^k f \qquad Y_k = (-1)^k\left(u^k - v^k\right)h.$$

Proof. The following relations hold in $\widehat{\mathfrak{OA}}$ for $n \geqslant 0$:

$$\sum_{k\geqslant 0}\binom{n}{k}X_k = A_n = 2t^n e + 2t^{-n} f$$

$$\sum_{k\geqslant 0}\binom{n}{k}(-1)^k Y_k = G_n = (t^n - t^{-n})h.$$

By an induction procedure, our results follow from the identities

$$t^n = (u + 1)^n = \sum_{k\geqslant 0}\binom{n}{k}u^k$$

$$t^{-n} = (v + 1)^n = \sum_{k\geqslant 0}\binom{n}{k}v^k. \qquad \square$$

Using lemma 10 and proposition 3, one obtains the following result.

Proposition 4. $\mathfrak{OA}_L$ *is a solvable Lie algebra of dimension* $L + [L/2]$ *with a basis consisting of* X_k *and* Y_j *with* $0 \leqslant k, j < L$ *and* j *odd.*

Let us examine the structure of the quotient $\mathfrak{OA}_{2l}$ more closely. The quotient $\mathfrak{OA}_{2l}/[\mathfrak{OA}_{2l}, \mathfrak{OA}_{2l}]$ is one dimensional and is spanned by X_0. Following the recipe of Malcev [17] for studying the structure of solvable Lie algebras, we examine the spectrum of the operator ad X_0. The following result was found by looking in detail at the quotient algebras $\mathfrak{OA}_{2l}$ for small l and then was proved by direct calculation.

Lemma 11. *The operator* ad X_0 *has the eigenvalues* $0, \pm 4$ *on the space* $\mathfrak{D}\mathfrak{A}_{2l}$. *Each of the eigenspace is* l*-dimensional. A basis of each eigenspace is given as follows:*

$$0 : X_0, \sum_{k=2j-2}^{2l-1} (-1)^k \binom{k-1}{2j-4} X_k \qquad 2 \leqslant j \leqslant l$$

$$\pm 4 : 2Y_{2j-1} \pm \left(X_{2j-1} - \sum_{k=2j-1}^{2l-1} (-1)^k \binom{k-1}{2j-2} X_k \right) \qquad 1 \leqslant j \leqslant l.$$

Furthermore, the 0-eigenvectors commute with each other.

Unfortunately, the other commutation relations among the above eigenvectors are not so easy to determine. Therefore, we proceed in the following manner. Set

$$\begin{aligned} H_0 &= \tfrac{1}{2} X_0 \\ E_0 &= \tfrac{1}{4} Y_1 + \frac{1}{8} \left(X_1 - \sum_{k=1}^{2l-1} (-1)^k X_k \right) \\ F_0 &= \tfrac{1}{4} Y_1 - \tfrac{1}{8} \left(X_1 - \sum_{k=1}^{2l-1} (-1)^k X_k \right). \end{aligned} \tag{17}$$

Then they satisfy the relations

$$[H_0, E_0] = 2E_0 \qquad [H_0, F_0] = -2F_0.$$

Set

$$H_1 = [E_0, F_0]$$

and define inductively

$$E_{j+1} = \tfrac{1}{2}[H_1, E_j] \qquad F_{j+1} = -\tfrac{1}{2}[H_1, F_j] \qquad 0 \leqslant j \leqslant l-2. \tag{18}$$

Then we have

Lemma 12. $\left[E_{j+1}, F_{k-1}\right] = \left[E_j, F_k\right]$.

Proof. Using the definition of E_{j+1}, we have

$$\begin{aligned} [E_{j+1}, F_{k-1}] &= \left[\tfrac{1}{2}[H_1, E_j], F_{k-1}\right] \\ &= -\tfrac{1}{2}[[E_j, F_{k-1}], H_1] - \tfrac{1}{2}[[F_{k-1}, H_1], E_j]. \end{aligned}$$

Since $[E_j, F_{k-1}]$ belongs to the 0-eigenspace of ad X_0, the right-hand side in the above becomes

$$-\tfrac{1}{2}[E_j, [H_1, F_{k-1}]] = [E_j, F_k]. \qquad \square$$

By lemma 12, the commutator $[E_j, F_k]$ depends only on $j + k$, which we denote by $H_{j+k+1} := [E_j, F_k]$. By proposition 4, a basis of $\mathfrak{D}\mathfrak{A}_{2l}$ is given by X_j, $0 \leqslant j \leqslant 2l-1$ and Y_{2k+1}, $0 \leqslant k < l$. Through their relations with X_j, Y_j, one can see that H_j, E_j, F_j, $0 \leqslant j \leqslant l-1$ also constitute a basis of $\mathfrak{D}\mathfrak{A}_{2l}$. Furthermore, in view of the eigenvalue distribution of ad X_0, it is easy to see that E_js and F_js commute among themselves, respectively. Summarizing (17), (18) and lemma 12, we have

Theorem 4. *The elements E_j, F_j, H_j, $0 \leqslant j \leqslant l-1$ form a basis of the quotient algebra $\mathfrak{O}\mathfrak{A}_{2l}$, in which the following commutation relations hold:*

$$[E_j, F_k] = H_{j+k+1} \qquad [H_j, E_k] = 2E_{j+k} \qquad [H_j, F_k] = -2F_{j+k}$$
$$[E_j, E_k] = 0 \qquad [F_j, F_k] = 0$$

here if the indices exceed $l-1$ then the corresponding elements are regarded to be 0.

The above commutation relations suggest that the structure of $\mathfrak{O}\mathfrak{A}_{2l}$ is very close to $\big(\mathbb{C}[x]/x^l\mathbb{C}[x]\big) \otimes \mathfrak{sl}_2$. However, the actual structure differs slightly. To fix this we consider the formal algebra $\widehat{\mathfrak{O}\mathfrak{A}}$ and employ the loop representation of $\mathfrak{O}\mathfrak{A}$. Instead of the variables, $u = t-1$, $v = t^{-1}-1$, which we used before, a convenient coordinate system near $t = 1$ for our purpose now is the following one:

$$\lambda = \tfrac{1}{2}(t - t^{-1}).$$

One has $\mathbb{C}[[u]] = \mathbb{C}[[v]] = \mathbb{C}[[\lambda]]$. In fact, the relations of λ and u, v are given by

$$u = \lambda - 1 + \sqrt{1+\lambda^2}$$
$$v = -\lambda - 1 + \sqrt{1+\lambda^2}$$
$$\lambda = \frac{u(2+u)}{2(1+u)} = \frac{-v(2+v)}{2(1+v)}.$$

Using lemma 10 and substituting the above relations into (17), we have

$$H_0 = e + f$$
$$E_0 = \tfrac{1}{2}\lambda(-h + (e-f))$$
$$F_0 = \tfrac{1}{2}\lambda(-h - (e-f)).$$

We introduce the following elements in $\mathfrak{sl}_2$ (which correspond to Pauli matrices in the canonical representation of $\mathfrak{sl}_2$):

$$\sigma^1 = e + f \qquad \sigma^2 = -\sqrt{-1}e + \sqrt{-1}f \qquad \sigma^3 = h. \tag{19}$$

By the definition of E_j, F_j and H_j, we find that

$$H_j = \lambda^{2j}\sigma^1$$
$$E_j = \tfrac{1}{2}\lambda^{2j+1}(\sqrt{-1}\sigma^2 - \sigma^3)$$
$$F_j = \tfrac{1}{2}\lambda^{2j+1}(-\sqrt{-1}\sigma^2 - \sigma^3).$$

Using the automorphism of $\mathfrak{sl}_2$ by cyclic permuting σ^js, we obtain an isomorphism which gives the structure of $\mathfrak{O}\mathfrak{A}_{2l}$,

$$\mathfrak{O}\mathfrak{A}_{2l} \simeq \bigoplus_{j=0}^{l-1} \mathbb{C}\lambda^{2j}h + \bigoplus_{j=0}^{l-1} \mathbb{C}\lambda^{2j+1}e + \bigoplus_{j=0}^{l-1} \mathbb{C}\lambda^{2j+1}f. \tag{20}$$

Note that the above $\mathfrak{O}\mathfrak{A}_{2l}$ is a solvable Lie algebra of dimension $3l$. The derived ideal $[\mathfrak{O}\mathfrak{A}_{2l}, \mathfrak{O}\mathfrak{A}_{2l}]$ is a nilpotent ideal of dimension $3l-1$. The classification of nilpotent Lie algebras is, in general, known to be a wild problem. Santharoubane has proposed a programme of classifying nilpotent Lie algebras in [24]. By analysing the commutation relations of root vectors of nilpotent Lie algebras, Santharoubane associates a generalized Cartan matrix (GCM) to each nilpotent Lie algebra and reduced the classification problem of nilpotent Lie algebras to that of certain ideals in the nilpotent part of Kac–Moody algebras. However, this problem is not easy though. Santharoubane and others try to classify these ideals in classical Lie algebras. For affine Lie algebras the work is done for $A_1^{(1)}$ and $A_2^{(2)}$. According to his classification there are three series of nilpotent Lie algebras associated with $A_1^{(1)}$. One of these series denoted by $\mathcal{A}_{1,l-1,1}$ is isomorphic to

$$\mathbb{C}e + \bigoplus_{j=1}^{l-1} x^j \mathfrak{sl}_2 + \mathbb{C}x^l f. \tag{21}$$

We can give an explicit isomorphism between this Lie algebra and the derived ideal of $\mathfrak{O}\mathfrak{A}_{2l}$. However, since the formula is quite complicated, we will not give it here. Recall here that the affine Lie algebra $A_1^{(1)}$ has several realizations. The most well known one is the homogeneous realization

$$A_1^{(1)} \simeq (\mathbb{C}[x] \otimes \mathfrak{sl}_2) \oplus \mathbb{C}c.$$

There also exists the principal realization

$$A_1^{(1)} \simeq (\mathbb{C}[y^2, y^{-2}] \otimes h) \oplus (y\mathbb{C}[y^2, y^{-2}] \otimes e) \oplus (y\mathbb{C}[y^2, y^{-2}] \otimes f).$$

The presentation of the nilpotent Lie algebra (21) refers to the homogeneous realization, while the nilpotent Lie algebra appearing as the derived ideal of (20) refers to the principal realization of $A_1^{(1)}$. In conclusion, we found that the series of nilpotent Lie algebras $\mathcal{A}_{1,l,1}$ appears as the derived ideal of the quotient of the Onsager algebra $\mathfrak{O}\mathfrak{A}_{2l}$.

Now we go back to the structure problem of a general $\mathfrak{O}\mathfrak{A}_L$ and the formal algebra $\widehat{\mathfrak{O}\mathfrak{A}}$. In the following, we present a slightly different approach from the previous one by considering more on the loop structure of $\mathfrak{O}\mathfrak{A}$. First, we need the following relations of powers of the coordinates, u, v, λ.

Lemma 13. *In the Laurent series ring $\mathbb{C}((\lambda))$, for $n \in \mathbb{Z}$, we have*

$$u^{2n} + v^{2n}, u^{2n-1} + v^{2n-1} \in \lambda^{2n}\mathbb{C}[[\lambda^2]]$$
$$u^{2n} - v^{2n}, u^{2n+1} - v^{2n+1} \in \lambda^{2n+1}\mathbb{C}[[\lambda^2]].$$

In fact, the following ratios tend to 1 as $\lambda \to 0$

$$\frac{u^{2n} + v^{2n}}{2\lambda^{2m}} \qquad \frac{u^{2n-1} + v^{2n-1}}{(2n-1)\lambda^{2n}} \qquad \frac{u^{2n} - v^{2n}}{2n\lambda^{2n+1}} \qquad \frac{u^{2n+1} - v^{2n+1}}{2\lambda^{2n+1}}.$$

Now we derive the structure of $\widehat{\mathfrak{O}\mathfrak{A}}$, $\mathfrak{O}\mathfrak{A}_L$ as follows:

Theorem 5. *Denote the Lie subalgebras of $\mathfrak{sl}_2[[\lambda]]$ for $L \geqslant 0$*

$$\mathfrak{sl}_2\langle\langle\lambda\rangle\rangle := \mathbb{C}[[\lambda^2]]h + \lambda\mathbb{C}[[\lambda^2]]e + \lambda\mathbb{C}[[\lambda^2]]f \subset \mathfrak{sl}_2[[\lambda]]$$
$$\mathfrak{sl}_2\langle\langle\lambda\rangle\rangle^L := \mathfrak{sl}_2\langle\langle\lambda\rangle\rangle \bigcap \lambda^L \mathfrak{sl}_2[[\lambda]].$$

Then we have the isomorphism

$$\widehat{\mathfrak{O}\mathfrak{A}} \simeq \mathfrak{sl}_2\langle\langle\lambda\rangle\rangle$$

which induces the isomorphisms

$$\widehat{\mathfrak{O}\mathfrak{A}}^L \simeq \mathfrak{sl}_2\langle\langle\lambda\rangle\rangle^L \qquad \mathfrak{O}\mathfrak{A}_L \simeq \mathfrak{sl}_2\langle\langle\lambda\rangle\rangle/\mathfrak{sl}_2\langle\langle\lambda\rangle\rangle^L.$$

Proof. With the elements σ^js (19) of $\mathfrak{sl}_2$ as before, there is a natural isomorphism

$$\mathfrak{sl}_2\langle\langle\lambda\rangle\rangle \simeq \mathfrak{sl}_2\langle\langle\lambda\rangle\rangle' := \mathbb{C}[[\lambda^2]]\sigma^1 + \lambda\mathbb{C}[[\lambda^2]]\sigma^2 + \lambda\mathbb{C}[[\lambda^2]]\sigma^3.$$

We need only show the results by replacing $\mathfrak{sl}_2\langle\langle\lambda\rangle\rangle$ as $\mathfrak{sl}_2\langle\langle\lambda\rangle\rangle'$. The expressions of X_k and Y_k in lemma 10 become

$$X_k = (u^k + v^k)\sigma^1 + \sqrt{-1}(u^k - v^k)\sigma^2 \qquad Y_k = (-1)^k(u^k - v^k)\sigma^3$$

by which and relations in lemma 13, $\widehat{\mathfrak{O}\mathfrak{A}}$ is a subalgebra of $\mathfrak{sl}_2\langle\langle\lambda\rangle\rangle'$ and $\widehat{\mathfrak{O}\mathfrak{A}}^L \subseteq \mathfrak{sl}_2\langle\langle\lambda\rangle\rangle'^L$. There is an induced canonical morphism

$$\rho_L : \mathfrak{O}\mathfrak{A}_L = \widehat{\mathfrak{O}\mathfrak{A}}/\widehat{\mathfrak{O}\mathfrak{A}}^L \longrightarrow \mathfrak{sl}_2\langle\langle\lambda\rangle\rangle'/\mathfrak{sl}_2\langle\langle\lambda\rangle\rangle'^L$$

for $L \geqslant 0$. It remains to show that ρ_L is an isomorphism. By proposition 4, X_k, Y_{2j+1} ($0 \leqslant k, 2j+1 < L$) form a basis of $\mathfrak{O}\mathfrak{A}_L$. While in $\mathfrak{sl}_2\langle\langle\lambda\rangle\rangle'/\mathfrak{sl}_2\langle\langle\lambda\rangle\rangle'^L$, $\lambda^{2l}\sigma^1, \lambda^{2j+1}\sigma^2, \lambda^{2j+1}\sigma^3$, $0 \leqslant 2l, 2j+1 < L$ form a basis. By using the behaviour of ratios in lemma 13 near $\lambda = 1$, the matrix of ρ_L for these bases is an invertible lower triangular one. Hence ρ_L is an isomorphism. □

Remark. The isomorphism in the above theorem for $\mathfrak{O}\mathfrak{A}_L$ for $L = 2l$ is the one in (20). For $L = 2l + 1$, it is given by

$$\mathfrak{O}\mathfrak{A}_{2l+1} \simeq \bigoplus_{j=0}^{l} \mathbb{C}\lambda^{2j}h + \bigoplus_{j=0}^{l-1} \mathbb{C}\lambda^{2j+1}e + \bigoplus_{j=0}^{l-1} \mathbb{C}\lambda^{2j+1}f$$

which can be derived by the same argument as before for (20) by examining the eigenvectors of $\mathrm{ad}X_0$.

6. Irreducible representations of the Onsager algebra and the superintegrable chiral Potts model

In this section, we are going to derive the classification of irreducible representations of the Onsager algebra. For $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{C}^{*n}$, one has the evaluation morphism of $\mathfrak{O}\mathfrak{A}$ into the sum of n copies of $\mathfrak{sl}_2$ defined by

$$ev_{\boldsymbol{a}} : \mathfrak{O}\mathfrak{A} \longrightarrow \bigoplus^{n} \mathfrak{sl}_2 \qquad X \mapsto (ev_{a_1}(X), \ldots, ev_{a_n}(X))$$

where ev_{a_j} is the evaluation of $\mathfrak{O}\mathfrak{A}$ at a_j. Denote

$$U_{\boldsymbol{a}}(t) := \prod_{a \in \{a_1, \ldots, a_n\}} U_a(t) \in \mathbb{C}[t].$$

Lemma 14. *We have* $\mathrm{Ker}(ev_{\boldsymbol{a}}) = \mathfrak{I}_{U_{\boldsymbol{a}}(t)}$. *The surjectivity of* $ev_{\boldsymbol{a}}$ *is equivalent to* $a_j \neq \pm 1, a_j \neq a_k^{\pm 1}$ *for* $j \neq k$.

Proof. For the determination of $\mathrm{Ker}(ev_{\boldsymbol{a}})$, through the diagonal map $\triangle : \mathfrak{sl}_2 \longrightarrow \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ and the involution $\theta : \mathfrak{sl}_2 \longrightarrow \mathfrak{sl}_2$, one may assume that $\boldsymbol{a} = (a_1, \ldots, a_n)$ satisfies the condition, $a_j \neq a_k^{\pm 1}$ for $j \neq k$. In fact, for simplicity let us take $n = 2$ as an example; a similar argument can apply to the case of a general n. When $a_1 = a_2 = a$, the evaluation map $ev_{(a,a)}$ of $\mathfrak{O}\mathfrak{A}$ can be reduced to ev_a by the relation $ev_{(a,a)} = \triangle ev_a$. When $a_1 = a_2^{-1} = a$, one reduces the map $ev_{(a,a^{-1})}$ to $ev_{(a,a)}$ by $ev_{(a,a^{-1})} = (\mathrm{id}, \theta)ev_{(a,a)}$. Hence both situations are reduced to the case $n = 1$. By lemma 4, the argument also shows the non-surjectivity of $ev_{\boldsymbol{a}}$ if $\boldsymbol{a}$ has two components with equal or reciprocal values; the same conclusion for $\boldsymbol{a}$ with one component equal to ± 1. Conversely, by lemma 4 $ev_{\boldsymbol{a}}$ is surjective for $\boldsymbol{a}$ with $a_j \neq \pm 1$ and $a_j \neq a_k^{\pm 1}$ for $j \neq k$. Now we may assume $\boldsymbol{a}$ with $a_j \neq a_k^{\pm 1}$. Then

$$\mathrm{Ker}(ev_{\boldsymbol{a}}) = \bigcap_{j=1}^{n} \mathrm{Ker}(ev_{a_j}) = \bigcap_{j=1}^{n} \mathfrak{I}_{U_{a_j}(t)} = \mathfrak{I}_{U_{\boldsymbol{a}}(t)}. \qquad \square$$

The quotient space of $\mathbb{C}^*$ by identifying a with a^{-1} is again parametrized by $\mathbb{C}^*$ with the variable denoted by $\hat{a} \in \mathbb{C}^*$, which is related to a by the rational map

$$\mathbb{C}^* \longrightarrow \mathbb{C}^* \qquad a \mapsto \hat{a} := \tfrac{1}{2}(a + a^{-1}).$$

By composing with the involution ι of $\mathfrak{O}\mathfrak{A}$, the representations $ev_{a^{-1}}$ and ev_a are equivalent $ev_{a^{-1}} = ev_a \iota$. We shall use the following convention if no confusion could arise:

$$\epsilon_{\hat{a}} := ev_a \qquad \epsilon_{\hat{\boldsymbol{a}}} := (\epsilon_{\hat{a}_1}, \ldots, \epsilon_{\hat{a}_n}) \qquad \text{for} \quad \hat{\boldsymbol{a}} = (\hat{a}_1, \ldots, \hat{a}_n).$$

Denote $\mathcal{S}$ the collection of all the non-trivial integral representations of $\mathfrak{sl}_2$. It is known that elements in $\mathcal{S}$ are labelled by positive half-integers s, which corresponds to the irreducible representation of $\mathfrak{sl}_2$ on the $(2s+1)$-dimensional vector space $V(s)$. Therefore, the effective irreducible integral representations of $\bigoplus^{n} \mathfrak{sl}_2$ are indexed by $\boldsymbol{s} = (s_1, \ldots, s_n) \in \mathcal{S}^n$, where $\bigoplus^{n} \mathfrak{sl}_2$ acts on the vector space $V(\boldsymbol{s}) (:= \otimes V(s_j))$ by the relation,

$$(x_1, \ldots, x_n)v = \sum_j (1 \otimes \cdots \otimes x_j \otimes \cdots \otimes 1)v \qquad x_j \in \mathfrak{sl}_2 \quad v \in V(\boldsymbol{s}).$$

Combining the above representation with $\epsilon_{\hat{\boldsymbol{a}}}$, one obtains a representation of $\mathfrak{O}\mathfrak{A}$ on $V(\boldsymbol{s})$

$$\rho_{(\hat{\boldsymbol{a}}, \boldsymbol{s})} : \mathfrak{O}\mathfrak{A} \longrightarrow \mathfrak{gl}(V(\boldsymbol{s})) \qquad (\hat{\boldsymbol{a}}, \boldsymbol{s}) \in \mathbb{C}^{*n} \times \mathcal{S}^n.$$

The Hermitian condition of $\rho_{(\hat{\boldsymbol{a}}, \boldsymbol{s})}$ is given by $|a_j| = 1$, equivalently, $\hat{a_j}$ in the real interval $[-1, 1]$ for all j, i.e. $\boldsymbol{a} = (\mathrm{e}^{\sqrt{-1}\theta_1}, \ldots, \mathrm{e}^{\sqrt{-1}}\theta_n)$, $\hat{\boldsymbol{a}} = (\cos(\theta_1), \ldots, \cos(\theta_n))$. Denote

$$\mathcal{C}_n := \{\hat{\boldsymbol{a}} = (\hat{a}_1, \ldots, \hat{a}_n) \in (\mathbb{C}^* \setminus \{\pm 1\})^n \mid \hat{a}_j \neq \hat{a}_k \text{ for } j \neq k\}$$
$$\mathcal{D}_n := \mathcal{C}_n \bigcap (-1, 1)^n.$$

By theorem 2 and the structure of $\mathrm{Ker}(ev_{\boldsymbol{a}})$ in lemma 14, one obtains the following results:

Proposition 5.

(I) $\mathrm{Ker}(\rho_{(\hat{\boldsymbol{a}}, \boldsymbol{s})}) = \mathfrak{I}_{U_{\boldsymbol{a}}(t)}$ *for* $(\hat{\boldsymbol{a}}, \boldsymbol{s}) \in \mathbb{C}^{*n} \times \mathcal{S}^n$.
(II) $\rho_{(\hat{\boldsymbol{a}}, \boldsymbol{s})}$ *is irreducible if and only if* $\hat{\boldsymbol{a}} \in \mathcal{C}_n$.
(III) $\rho_{(\hat{\boldsymbol{a}}, \boldsymbol{s})}$ *is irreducible Hermitian if and only if* $\hat{\boldsymbol{a}} \in \mathcal{D}_n$.
(IV) For $(\hat{\boldsymbol{a}}, \boldsymbol{s}) \in \mathcal{C}_n \times \mathcal{S}^n$, $(\hat{\boldsymbol{a}}', \boldsymbol{s}') \in \mathcal{C}_{n'} \times \mathcal{S}^{n'}$, *the representations* $\rho_{(\hat{\boldsymbol{a}}, \boldsymbol{s})}$, $\rho_{(\hat{\boldsymbol{a}}', \boldsymbol{s}')}$ *are equivalent if and only if* $n = n'$ *and* $\hat{a}'_j = \hat{a}_{\sigma(j)}$, $s'_j = s_{\sigma(j)}$ *for some permutation* σ *of indices.*

Now we are going to classify all the irreducible representations of $\mathfrak{O}\mathfrak{A}$.

Lemma 15. *Let* ρ *be a non-trivial irreducible representation of* $\mathfrak{O}\mathfrak{A}$ *in* $\mathfrak{sl}(V)$. *Then* $\mathrm{Ker}(\rho) = \mathfrak{I}_{U_{\boldsymbol{a}}(t)}$ *for some* $\boldsymbol{a} = (a_1, \ldots, a_n) \in (\mathbb{C}^* \setminus \{\pm 1\})^n$ *with* $a_j \neq a_k^{\pm 1}$ *for* $j \neq k$.

Proof. By Schur's lemma, $\mathrm{Ker}(\rho)$ is a closed ideal. By theorem 2, $\mathrm{Ker}(\rho) = \mathfrak{I}_{P(t)}$ for some $P(t) = \prod_{j=1}^n U_{a_j}(t)^{m_j}$, where $a_j \in \mathbb{C}^*, m_j \in \mathbb{Z}_{>0}$ with $a_j \neq a_k^{\pm 1}$ for $j \neq k$, and m_j even whenever $a_j = \pm 1$. It suffices to show $m_j = 1$ for all j. Otherwise, we may assume $m_1 \geqslant 2$. Define the polynomial $R(t) = U_{a_1}(t)^{m_1-1} \prod_{j>1} U_{a_j}(t)^{m_j}$. Then $\mathfrak{I}_{R(t)}/\mathfrak{I}_{P(t)}$ is a non-trivial Abelian ideal of $\mathfrak{OA}/\mathfrak{I}_{P(t)}$. By the irreducibility of ρ, one has $V = (\mathfrak{I}_{R(t)}/\mathfrak{I}_{P(t)})V$. Let $V = V_1 \oplus \cdots \oplus V_r$ be the eigenspace decomposition of V with respect to the $\mathfrak{I}_{R(t)}/\mathfrak{I}_{P(t)}$ action, then eigenvalues λ_j for V_j $(1 \leqslant j \leqslant r)$ are distinct. Here λ_j is a linear functional on $\mathfrak{I}_{R(t)}/\mathfrak{I}_{P(t)}$. As the representation takes value in $\mathfrak{sl}(V)$, the number r is at least 2. We are going to show that the vector space V_1 gives rise to a subrepresentation of $\mathfrak{OA}$, hence a contradiction to the irreducibility of ρ. Let v be an element of V_1 and $X \in \mathfrak{OA}$. Denote $\rho(X)(v) = \sum_{l=1}^r v_l$ with $v_j \in V_j$. For each $j \geqslant 2$, we choose an element $Z_j \in \mathfrak{I}_{R(t)}$ such that the class of Z_j in $\mathfrak{I}_{R(t)}/\mathfrak{I}_{P(t)}$ takes different values for λ_1 and λ_j, $\lambda_1(Z_j + \mathfrak{I}_{P(t)}) \neq \lambda_j(Z_j + \mathfrak{I}_{P(t)})$. As $\rho(X)\rho(Z_j) - \rho(Z_j)\rho(X)$ $(= \rho([X, Z_j])$ is an element of $\rho(\mathfrak{I}_{R(t)})$, we have

$$V_1 \ni \rho(X)\rho(Z_j)(v) - \rho(Z_j)\rho(X)(v) = \lambda_1(Z_j + \mathfrak{I}_{P(t)}) \sum_{l=1}^r v_l - \sum_{l=1}^r \lambda_l(Z_j + \mathfrak{I}_{P(t)}) v_l$$

which implies $v_j = 0$ for $j \geqslant 2$. Therefore, V_1 is a representation of $\mathfrak{OA}$. □

Now we can derive the following result in [9, 23].

Theorem 6. *Any non-trivial irreducible representation of $\mathfrak{OA}$ is represented by $\rho_{(\hat{a},s)}$ for some $(\hat{a}, s) \in \mathcal{C}_n \times \mathcal{S}^n$, $n \in \mathbb{Z}_{>0}$. Subsequently, all the irreducible Hermitian representations of $\mathfrak{OA}$ are given by $\rho_{(\hat{a},s)}$ for $(\hat{a}, s) \in \bigsqcup_{n\in\mathbb{Z}_{>0}} (\mathcal{D}_n \times \mathcal{S}^n)$, modulo the following relation:*

$$(\hat{a}, s) = ((\hat{a}_1, \ldots, \hat{a}_n)), (s_1, \ldots, s_n)) \sim (\hat{a}', s') = ((\hat{a}'_1, \ldots, \hat{a}'_n)), (s'_1, \ldots, s'_n))$$

where $\hat{a}'_j = \hat{a}_{\sigma(j)}$, $s'_j = s_{\sigma(j)}$ and σ is a permutation of indices.

Proof. For a non-trivial irreducible representation ρ of $\mathfrak{OA}$, by lemmas 1, 4 and 15, the Lie-algebra $\mathfrak{OA}/\mathrm{Ker}(\rho)$ is isomorphic to $\overset{n}{\oplus}\, \mathfrak{sl}_2$ for some positive integer n. As an irreducible representation of $\overset{n}{\oplus}\, \mathfrak{sl}_2$ is obtained by tensoring irreducible one of its factors, the result follows immediately. □

Remark. As is known in the affine Lie algebra theory, all irreducible finite-dimensional representations of $\mathfrak{sl}_2$-loop algebra (or equivalently $\mathfrak{sl}_2$-affine algebra) are isomorphic to tensor products of irreducible representations of $\mathfrak{sl}_2$ through evaluation at distinct non-zero complex number a_js (see, e.g., [13], exercise 12.11). While for the irreducible representations of Onsager algebra $\mathfrak{OA}$, one requires all the values $a_j \neq \pm 1$, plus the identification of a_j and a_j^{-1} which produce the same representation of $\mathfrak{OA}$. Hence the discussion of section 5 on the effect when the Onsager algebra is valued at ± 1 reveals the essence of the Onsager algebra different from the $\mathfrak{sl}_2$-loop algebra from the representation point of view.

We now discuss a physical application of the previous results to the superintegrable chiral Potts N-state model. The Hamiltonian is the spin chain of a parameter k' [2–7, 12, 20],

$$H(k') = H_0 + k' H_1$$

where H_0 and H_1 are Hermitian operators acting on the vector space of the L-tensor of $\mathbb{C}^N$, defined by

$$H_0 = -2\sum_{l=1}^{L}\sum_{n=1}^{N-1}(1-\omega^{-n})^{-1}X_l^n \qquad H_1 = -2\sum_{l=1}^{L}\sum_{n=1}^{N-1}(1-\omega^{-n})^{-1}Z_l^n Z_{l+1}^{N-n}$$

where $\omega = \mathrm{e}^{(2\pi\sqrt{-1})/N}$, $X_l = I\otimes\cdots\otimes \overset{l\text{th}}{X}\otimes\cdots\otimes I$, $Z_l = I\otimes\cdots\otimes \overset{l\text{th}}{Z}\otimes\cdots\otimes I$ $(Z_{L+1} = Z_1)$. Here I is the identity operator, X and Z are the operators of $\mathbb{C}^N$ with the relation, $ZX = \omega XZ$, defined by $X|m\rangle = |m+1\rangle$, $Z|m\rangle = \omega^m|m\rangle$, $m\in\mathbb{Z}_N$. The operator $H(k')$ is Hermitian for real k', hence with the real eigenvalues. It is the Ising quantum chain [14] for $N = 2$. For $N = 3$, one obtains the $\mathbb{Z}_3$-symmetrical self-dual chiral clock model with the chiral angles $\pi/2$, which was studied by Howes *et al* [16]. For general N, we set

$$A_0 = -2N^{-1}H_0 \qquad A_1 = -2N^{-1}H_0$$

then A_0 and A_1 satisfy the DG condition, which by theorem 6 and proposition 5, ensures that the eigenvalues of the unitary operator $H(k')$ have the following special form as in the Ising model:

$$a + bk' + 2N\sum_{j=1}^{n} m_j\sqrt{1+k'^2-2k'\cos(\theta_j)}$$

where $a, b, \theta_j \in \mathbb{R}$ and $m_j = -s_j, (-s_j+1), \ldots, s_j$, with s_j being a positive half-integer [9]. For the ground-state sectors, by the computation of the superintegrable chiral Potts model Baxter [5] obtained the corresponding eigenvalue of $H(k')$ given by the spin-$\frac{1}{2}$ representation of $\mathfrak{sl}_2$, i.e. $s_j = \frac{1}{2}$ for all j, with an explicit formula of θ_js (see, e.g., [23]). However, these results are not obvious from the representation theoretic point of view. The understanding of the exact form of eigenvalues of $H(k')$ has still been left as a theoretical challenge of the study of Onsager algebra.

7. Further remarks

In this paper, we have obtained the structure of closed ideals I of the Onsager algebra $\mathfrak{OA}$ and established their relation with reciprocal polynomials $P(t)$, $I = \mathfrak{I}_{P(t)}$. For $P(t) = U_a(t)^L$, we have determined the Lie algebra structure of the quotient algebra $\mathfrak{OA}/\mathfrak{I}_{P(t)}$. These results, together with a polynomial $P(t)$ of mixed types, should make some significant extensions of our knowledge of solvable or nilpotent algebras. Generalizations of the Onsager algebra to other loop groups or Kac–Moody algebras, like that in [1, 25], should provide ample examples of solvable algebras. The intimate relation of Onsager algebra and the superintegrable chiral Potts model described in section 6 also suggests the potential links of Onsager algebra or its generalized ones to other quantum integrable systems in statistical mechanics. We hope that the further development of the subject will eventually lead to some interesting results in Lie-theory with possible applications in quantum integrable models.

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